$$\Delta\delta_f = \frac{\delta_f - \tilde{\delta}_f}{\delta_f}, \quad (12)$$

where $\delta_f$ is the exact value of the error; $\tilde{\delta}_f$ is the value known with an error. The results demonstrated show that the overdetermination of the level of error results in more appreciable errors in the solution of the inverse heat conduction problem in the metrics $C^0$ in comparison with the underdetermination. In comparing the accuracy $\delta_u$ in the metrics $L_2$, overdetermination and underdetermination in $\delta_f$ results in approximately equal results.

## NOTATION

T, temperature; C, volumetric heat capacity; λ, thermal conductivity; f, additional temperature measurement; q, heat flux density to the internal body surface; u, heat flux density to the external surface; $\delta_f$, measurement error; $\tau_{max}$, duration of process; b, body thickness.

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# DEGREE OF INSTABILITY OF NUMERICAL SOLUTIONS OF INVERSE HEAT-CONDUCTION PROBLEMS AND ERROR OF EXPERIMENTAL DATA


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UDC 536.6



A method is proposed for estimating the error of the results obtained in analyzing experimental data using the solutions of nonsteady boundary inverse heat-conduction problems.


An important aspect of the applied use of the solution of inverse heat-conduction problems is the question of determining the error of the results obtained [1-3]. In the present work, a solution of this problem is proposed for a sufficiently broad class of nonsteady boundary inverse heat-conduction problems in a linear formulation, expressed as an integral equation

$$\int_0^{\tau} q(t)\, G(\tau - t)\, dt = T_{\delta}(\tau) - T_0, \quad (1)$$

where $T_{\delta}(\tau)$ is the temperature dependence, measured with an error of $\delta T$; $T_0$ is the initial temperature; $G(\tau) = (\partial/\partial\tau)[U(\tau)]$.

Solution of Eq. (1) by approximating the desired heat flux as a piecewise-constant function (direct algebraic method [1]) is expressed by the following recurrence relation

$$q_i = \frac{1}{G_1}\left(T_i - T_0 - \sum_{j=1}^{i-1} q_j G_{i-j+1}\right), \quad i = 1,\ 2,\ \ldots,\ m. \quad (2)$$

Here $T_i = T_{\delta}(\tau_i)$, $\tau_i = i\Delta\tau$; $q_i$ is the heat flux in the i-th time interval $(\tau_{i-1}, \tau_i)$.


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Now Eq. (2) is transformed using a discrete analog of Eq. (1) in the form

$$\sum_{j=1}^{i-1} q_j G_{i-j} = T_{i-1} - T_0. \tag{3}$$

Combining Eqs. (2) and (3) allows the solution of the given problem to be written in the form

$$q_i = \frac{1}{G_1}(T_i - T_{i-1}) + \sum_{j=1}^{i-1} q_j \varphi_{i-j}, \quad i = 1, 2, \dots, m;$$
$$\varphi_i = \frac{G_i - G_{i+1}}{G_1}, \quad i = 1, 2, \dots, m-1. \tag{4}$$

Equation (4) determines the dependence of the errors of the solutions $q_i$ obtained on the error of the temperature measurements $\delta T_i$

$$\delta q_i = \frac{1}{G_1}(\delta T_i - \delta T_{i-1}) + \sum_{j=1}^{i-1} \delta q_j \varphi_{i-j}. \tag{5}$$

Introducing the greatest deviation of the given dependences from the true values as a measure of their error

$$\delta q = \sup_i |\delta q_i|; \quad \delta T = \sup_i |\delta T_i|, \tag{6}$$

the first term in Eq. (5) may be written in the following form, in the chosen notation

$$\delta q^* = k\frac{\delta T}{G_1}. \tag{7}$$

The value of the constant k is determined by the character of the distribution of the error $\delta T$. Under the assumption that all the $\delta T_i$ are independent normally distributed random quantities, $k = \sqrt{2}$.

Whereas the level of error determined by the first term in Eq. (5) is independent of the number of the time step i, this dependence becomes significant in the second term

$$\delta q_i^{**} = \sum_{j=1}^{i-1} \delta q_j^{**} \varphi_{i-j}, \quad i = 2, 3, \dots, m. \tag{8}$$

As shown by Eq. (8), any perturbation – for example, $\delta q_1^{**}$ – may grow either weaker or stronger in the course of calculation.

As a quantitative characteristic of this feature of the computational process, the parameter $\beta$ may be introduced; below, it is called the degree of instability of numerical solution of the inverse heat-conduction problem

$$\delta q = \beta \delta q^*. \tag{9}$$

The parameter $\beta$ introduced in this way shows to what extent the level of error $\delta q^*$ defined according to Eq. (7) is intensified in the course of numerical realization of the algorithm.

Using Eq. (8), a sequence of obvious inequalities may be obtained

$$|\delta q_i^{**}| \leqslant \sum_{j=1}^{i-1} |\delta q_j^{**}| |\varphi_{i-j}| \leqslant \delta q_s^{**} \sum_{j=1}^{i-1} |\varphi_{i-j}|, \quad \delta q_s^{**} = \sup_i |\delta q_i^{**}|. \tag{10}$$

Analysis of Eq. (10) shows that if

$$\eta = \sum_{j=1}^{i-1} |\varphi_j| \leqslant 1 \tag{11}$$

any perturbation – in particular, $\delta q_1^{**}$ – is not intensified in the corresponding calculations, i.e., the following sequence of inequalities holds

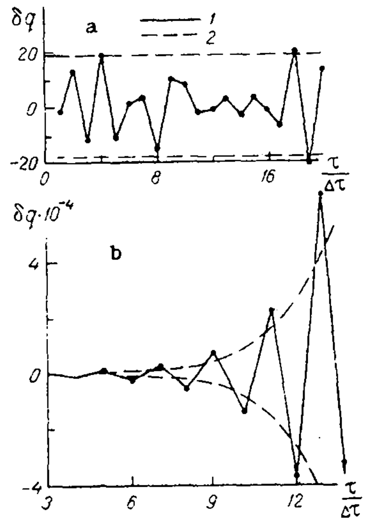


Fig. 1

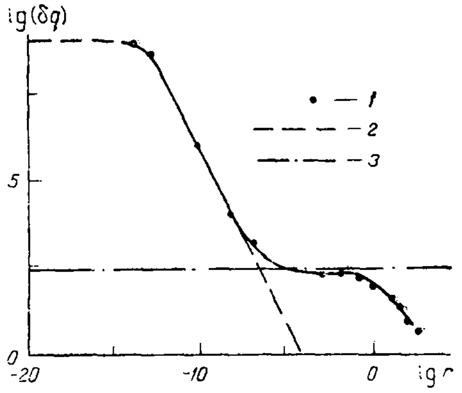


Fig. 2

Fig. 1. Error of nonregularized solutions of the inverse heat-conduction problems; a) stable solution; $\Delta Fo = a\Delta\tau/x^2 = 0.5$; $\delta T = 2\sigma T$; $\varphi_1 = 0.4$; 1) numerical calculation; 2) estimate from Eq. (7); b) unstable solution; $\Delta Fo = 0.2$; $\varphi_1 = -1.5$; 1) numerical calculation; 2) estimate according to Eqs. (9) and (17).

Fig. 2. Influence of regularization on the error of solution of the inverse heat-conduction problem ($\Delta Fo = 0.1$; $\varphi_1 = -5.7$; $\beta \sim 4\cdot10^6$); 1) numerical calculation; 2) estimate from Eq. (20); 3) from Eq. (7).

$$|\delta q_1^{**}| \geqslant |\delta q_2^{**}| \geqslant \ldots \geqslant |\delta q_m^{**}|. \tag{12}$$

Taking account of the form of the pulsed characteristics of the given thermal processes [1], the condition in Eq. (11) may be written in the form

$$\eta = \frac{1}{G_1}(2G_s - G_1 - G_m) \leqslant 1, \tag{13}$$

where $G_s = \sup_i G_i$.

It is readily evident that, in the limiting case ($G_m \to \infty$, $m \to \infty$), the condition in Eq. (13) reduces to the well-known stability condition of the computational process, which takes the form $G_2 = G_1$ in the present notation [1].

It is clear from these relations that, if Eq. (13) holds, then $\beta \sim 1$, i.e., the error of the solutions of the inverse heat-conduction problems obtained here may be estimated from Eq. (7). Numerical experiments show that, instead of Eq. (13), a weaker stability condition of the given algorithm may be used

$$|\varphi_1| \leqslant 1, \tag{14}$$

which is practically equivalent to the condition $\Delta F_0 \geq 0.31$ for the most-studied case of a semiinfinite wall [4]. Otherwise, i.e., if

$$|\varphi_1| > 1, \tag{15}$$

the numerical algorithm obtained may be assigned to the unstable category. For a sufficiently broad class of practical problems determined by the condition

$$|\varphi_1| > |\varphi_i|, \quad i = 2, 3, \ldots, m-1, \tag{16}$$

the degree of instability of the numerical solutions may be estimated as follows

$$\beta \sim |\varphi_1|^m. \quad (17)$$

Characteristic results of the numerical experiment are shown in Fig. 1. It is evident from these data that the use of the direct algebraic method when Eq. (15) holds is irrational from the viewpoint of the error of the results obtained. An effective method of solution of such problems is the Tikhonov regularization method [1, 3], which consists in minimization of the following regularizing functional

$$\Phi(q) = \| Aq - T_\delta \|^2_{L_2} + \alpha \| q' - q'_* \|^2_{L_2}, \quad (18)$$

where Aq is the algorithm for solution of the corresponding direct problem; $q_*$ is the trial solution; $\alpha$ is the regularization parameter.

To elucidate the question of the error solutions obtained by the regularization method [1-3], which has not been adequately studied as yet, a numerical experiment is undertaken. It indicates the possibility of significant simplification of the given problem by introducing the dimensionless regularization parameter in the form

$$r = \frac{\alpha}{(G_1 \Delta\tau)^2}. \quad (19)$$

In this case, the error of the regularized solutions obtained $\delta q_0$ may be estimated when $0 \le r \le 1/\beta$ as follows

$$\delta q = \frac{\delta q_0}{1 + r\beta^2}, \quad (20)$$

where $\delta q_0$ is the error of the unregularized solution, i.e., the solution obtained when r = 0. Note that Eq. (20) is obtained by generalizing the calculation results for the case when numerical realization of the given algorithm is not associated with marked influence of the rounding error in the calculations on the error of the results obtained. This is easily established from the coincidence of the initial dependence of $T_\delta$ and the inverse dependence obtained by solution of the direct problem with respect to the solution obtained when r = 0, i.e., by verifying the obvious equation

$$Aq_{r=0} = T_\delta. \quad (21)$$

In the range of variation of the dimensionless regularization parameter

$$\frac{1}{\beta} \leqslant r \leqslant 1 \quad (22)$$

the error of the solutions obtained changes little and may be estimated from Eq. (7). This result indicates that, in the range in Eq. (22), there is a quasi-optimal value of r in weakened form [3].

The results of numerical experiment confirming the validity of the above relations are shown in Fig. 2 for the case of a semiinfinite wall ($\Delta$Fo = 0.1), which corresponds to $\beta \sim 4 \cdot 10^6$ when m = 10.

Thus, the use of a quasi-optimal regularization parameter in weakened form is essentially based on the idea that the desired solution does not contain an oscillatory component with a characteristic time $\tau_0 = 2\Delta\tau$ and an amplitude that increases over time.

Decrease in the error below the level in Eq. (7), which is possible when r > 1, requires the use of more complete a priori information on the desired solution.

## NOTATION

$\tau$, time; $T(\tau)$, temperature; $q(\tau)$, heat flux; $\delta T$, $\delta q$, errors in determining the temperature and heat flux; $G(\tau)$, Green's function; $U(\tau)$, solution of direct heat-conduction problem with $q(\tau) = 1$; $\Delta\tau$, time step; x, depth of thermocouple; $a$, thermal diffusivity; $\beta$, degree of instability of numerical solution of inverse heat-conduction problem; r, dimensionless regularization parameter; $\varphi$, defined in Eq. (4). Indices: 0, initial value; i, j = 1, 2, ..., m, number of time interval.

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# A NONLINEAR REGULARIZING ALGORITHM FOR SOLVING ONE CLASS OF INVERSE PROBLEMS OF HEAT CONDUCTION


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UDC 519.2


A nonlinear regularizing algorithm is proposed for solving ill-conditioned systems of equations, which takes account of typical a priori information about the solution sought.


Numerous formulations of the inverse heat conduction problem [1] (specifically, a parametrized identification of heat conduction processes) lead to the system of linear algebraic equations of the form

$$K\varphi = f \tag{1}$$

As a rule, system (1) is ill-conditioned or degenerate and in order to construct a stable (regularized) solution, different methods of regularization of the solution are used [2, 3]. In this work, a method is presented for constructing a regularized solution on the basis of the singular expansion of the matrix K, taking into account preliminary information that is typical for the considered problem.

Linear Regularizing Algorithm. We assume for definiteness that the matrix K is of order $N_f \times N_\varphi$, where $\varphi$ and f are vectors of appropriate dimensionality. The representation [4] $K = U\Lambda V^T$ is called a singular expansion of the matrix K, in which U and V are orthogonal matrices of order $N_f \times N_f$, $N_\varphi \times N_\varphi$, where T is the transpose sign, $\Lambda$ is a matrix of order $N_f \times N_\varphi$ with elements

$$\{\Lambda\}_{i,j} = \begin{cases} \lambda_i, & i = j; \\ 0, & i \neq j. \end{cases}$$

The values $\lambda_i \geq 0$, i = 1, 2, ..., $N_\varphi$, are called singular numbers of the matrix K. Suppose that a) $N_j \geq N_\varphi$; b) singular numbers are ordered: $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_{N_\varphi} \geq 0$; c) instead of the exact right-hand side of f, the vector $\tilde{f} = f + \eta$, where $\eta$ is a random vector with zero mean, reflecting errors in specifying the right-hand side of Eq. (1), is specified.

A solution $\varphi_\alpha$ stable with respect to noise $\eta$ and to the errors of the computational process realizing a singular expansion can be represented in the following form

$$\begin{gathered} \varphi_\alpha = Vx_\alpha, \\ x_\alpha(j) = r_\alpha(j)\tilde{y}(j), \quad r_\alpha(j) = 1/(\lambda_j + \alpha m(\lambda_j)), \quad 1 \leqslant j \leqslant N_\varphi, \end{gathered} \tag{2}$$

where $x_\alpha$ is an $N_\varphi$-dimensional vector; $x_\alpha(j)$ is its j projection; $\tilde{y}(j)$ is the j-projection of the vector $\tilde{y} = U^T f$; $\alpha$ is a parameter of regularization; $m(\lambda)$ is a nonincreasing positive function (for example, $m(\lambda) = \lambda^{-\theta}$, $\theta \geq 1$). It can be shown that for an appropriate choice of $\alpha$ the solution $\varphi_\alpha$ is regularized, i.e., when the errors tend to zero, $\varphi_\alpha$ converges to the exact pseudosolution of system (1). By not considering the choice of $\alpha$, we only note that existing algorithms for estimating an optimal (in the sense of a root-mean-square


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